$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

RHS

$f(x) = x \rightarrow f'(x) = 1$

$g'(x) \rightarrow g(x)$

ECO106/138 Week 2 Lecture Note Template

Integration by Substitution: Indefinite Integral

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du, \quad u = g(x) \Rightarrow \frac{du}{dx} = g'(x)$$

$$du = g'(x)\,dx$$

$$\int f(u)\,du$$

e.g. $\int (x^2+10)^{50} 2x dx = \int u^{50}\,du = \frac{1}{51}u^{51} + c$

$= \frac{1}{51}(x^2+10)^{51} + c$ //

$u = x^2 + 10$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

Try to replace all xs with u

e.g. $\int 8x^2(3x^3-1)^{16}dx = \frac{8}{9}\int u^{16}\,du = \frac{8}{9}\left[\frac{1}{17}u^{17} + c\right]$

$= \frac{8}{9}du$

$= \frac{8}{153}(3x^3-1)^{17} + c$ //

$\rightarrow u = 3x^3 - 1$

$\frac{du}{dx} = 9x^2 \Rightarrow du = 9x^2dx$

$\frac{8}{9}du = \frac{8}{9}(9x^2dx)$

$= 8x^2dx$

e.g. $\int \frac{\ln x}{\sqrt{x}}dx = \int \frac{2\ln u}{\cancel{u}}\, 2\cancel{u}\,du = \overset{4}{\cancel{2}}\int \ln u\,du = \overset{4}{\cancel{2}}[u\ln u - u] + c$

$u = \sqrt{x} = x^{1/2} \Longrightarrow \boxed{u^2 = x} \Rightarrow \frac{dx}{du} = 2u \Rightarrow dx = 2u\,du$

$\ln x = 2\ln u$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$du = \frac{1}{2\sqrt{x}}dx$

$2du = \frac{1}{\sqrt{x}}dx$

$\ln x$?

$= \overset{4}{\cancel{2}}\sqrt{x}\,\underbrace{\ln\sqrt{x}}_{=\frac{1}{2}\ln x} - 4\sqrt{x} + c$

$= 2\sqrt{x}\ln x - 4\sqrt{x} + c$

$= 2\sqrt{x}\ln x - 4\sqrt{x} + c$ //

Integration by Substitution : Definite Integral

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \quad , \qquad u = g(x)$$

e.g. $\int_1^e \frac{1+\ln x}{x}dx = \int_{1+\ln 1}^{1+\ln e} u\,du = \int_1^2 u\,du = \frac{1}{2}u^2\Big|_1^2 = \frac{1}{2}[2^2 - 1^2] = \frac{3}{2}$ //

$\boxed{u = 1 + \ln x}$

$u = 1 + \ln(1)$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}dx$

e.g. $\int_0^4 \frac{1}{\sqrt{1+\sqrt{x}}}dx = \int_{\sqrt{1+\sqrt{0}}}^{\sqrt{1+\sqrt{4}}} 4(u^2-1)\,du = \int_1^{\sqrt{3}} 4(u^2-1)\,du$

$\rightarrow u = \sqrt{1+\sqrt{x}} \rightarrow u^2 = 1+\sqrt{x} \rightarrow \boxed{\sqrt{x} = u^2-1}$

$\rightarrow \frac{du}{dx} = \frac{1}{2}(1+\sqrt{x})^{-\frac{1}{2}}\frac{1}{2}x^{-1/2}$

$du = \frac{1}{4}\frac{1}{\sqrt{1+\sqrt{x}}\,\sqrt{x}}dx$

$4\sqrt{x}\,du = \frac{1}{\sqrt{1+\sqrt{x}}}dx$

$4(u^2-1)\,du =$

$= \frac{4}{3}u^3 - 4u \Big|_1^{\sqrt{3}}$

$= \left(\frac{4}{3}3\sqrt{3} - 4\sqrt{3}\right) - \left(\frac{4}{3} - 4\right)$

$= 4 - \frac{4}{3}$

$= \frac{12-4}{3} = \frac{8}{3}$ //

e.g. $\int_1^3 \frac{1}{x^2}e^{2/x}dx = -\frac{1}{2}du$

$= \int_{\frac{2}{1}}^{\frac{2}{3}} -\frac{1}{2}e^u\,du$

$u = \frac{2}{x} = 2x^{-1}$

$\frac{du}{dx} = -2x^{-2} = \frac{-2}{x^2}$

$du = -\frac{2}{x^2}dx$

$-\frac{1}{2}du = -\frac{1}{2}\left(\frac{-2}{x^2}dx\right)$

$= \frac{1}{x^2}dx$

$= - \int_{2/3}^{2}(-\frac{1}{2})e^u\,du$

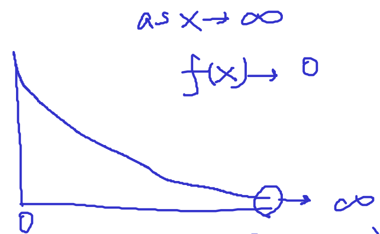
$= \frac{1}{2}\int_{2/3}^{2} e^u\,du$

$= \frac{1}{2}\left[e^u \Big|_{2/3}^{2}\right]$

$= \frac{1}{2}\left[e^2 - e^{2/3}\right]$

$= 2.72$ //

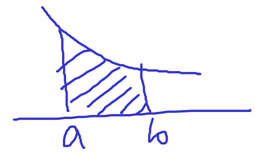
Infinite Intervals of Integration



f: continuous for all $x \geq a$.

$\int_a^b f(x)dx$ is defined for b$\geq a$.

If the limit of $\int_a^b f(x)dx$ as b -> ∞ exists, f is integrable over [a ∞).



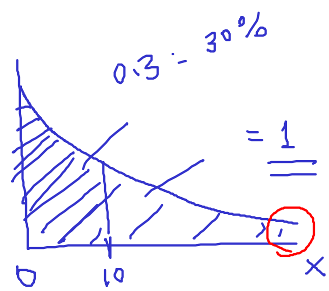
$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \boxed{\int_a^b f(x)dx}$$

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$$

*If the limit doesn't exist, improper integral is said to diverge.

e.g. $f(x) = \lambda e^{-\lambda x}, x \geq 0, \lambda \geq 0$ $(Exponential\ Distribution)$

Show the area below the graph of f over [0 ∞) = 1.



$$\int_0^{\infty} \lambda e^{-\lambda x}\, dx$$

$$= \lim_{b\to\infty} \int_0^b \lambda e^{-\lambda x}\, dx \qquad \frac{\lambda}{-\lambda} e^{-\lambda x} = -e^{-\lambda x}$$

$$= \lim_{b\to\infty} -e^{-\lambda x} \Big|_0^b = \lim_{b\to\infty} \left[-e^{-\lambda b} + e^0\right] = \lim_{b\to\infty} \left(-e^{-\lambda b} + 1\right)$$

$$e^{-\lambda b} = \frac{1}{e^{\lambda b}} \to 0$$

$$= 1$$

If both limits are infinites:

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} \underbrace{f(x)\,dx} + \int_{0}^{\infty} \underbrace{f(x)\,dx}}$$

e.g. $Solve\ \int_{-\infty}^{\infty} xe^{-\lambda x^2} dx,\ \lambda > 0.$

$$= \int_{-\infty}^{0} xe^{-\lambda x^2} dx + \int_{0}^{\infty} xe^{-\lambda x^2} dx$$

$$= \lim_{a \to -\infty} \underbrace{\int_{a}^{0} xe^{-\lambda x^2} dx}_{I_1} + \lim_{b \to \infty} \underbrace{\int_{0}^{b} xe^{-\lambda x^2} dx}_{I_2}$$

$$\begin{cases} u = -\lambda x^2 \\ \dfrac{du}{dx} = -2\lambda x \\ du = -2\lambda x\,dx \end{cases}$$

$$-\frac{1}{2\lambda} du = x\,dx$$

$$I_1 = \int_{-\lambda a^2}^{0} -\frac{1}{2\lambda} e^u du$$

$$= -\frac{1}{2\lambda} e^u \Big|_{-\lambda a^2}^{0}$$

$$= -\frac{1}{2\lambda}\left(e^0 - e^{-\lambda a^2}\right) \quad (e^{-\lambda a^2} \to 0)$$

$$\lim_{a \to -\infty} I_1 = -\frac{1}{2\lambda}$$

$$I_2 = \int_{0}^{-\lambda b^2} -\frac{1}{2\lambda} e^u du$$

$$= -\frac{1}{2\lambda} e^u \Big|_{0}^{-\lambda b^2}$$

$$= -\frac{1}{2\lambda}\left(e^{-\lambda b^2} - 1\right) \quad (e^{-\lambda b^2} \to 0)$$

$$\lim_{b \to \infty} I_2 = -\frac{1}{2\lambda}(-1) = \frac{1}{2\lambda}$$

$$I = \lim_{a \to -\infty} I_1 + \lim_{b \to \infty} I_2$$

$$= -\frac{1}{2\lambda} + \frac{1}{2\lambda} = 0$$

//

$I =$

e.g. Solve $\int_0^\infty x\lambda e^{-\lambda x} \; dx$. (Expected Value of Exponential Distribution)

= Average

$f(x) = x \qquad f'(x) = 1$

$g'(x) = \lambda e^{-\lambda x} \qquad g(x) = -e^{-\lambda x}$

$$I = \lim_{b \to \infty} \left(-x e^{-\lambda x} \Big|_0^b + \int_0^b + e^{-\lambda x} \, dx \right)$$

$$= \lim \left(-b e^{-\lambda b} + \left(\frac{1}{-\lambda} e^{-\lambda x} \Big|_0^b \right) \right)$$

$$= \lim \left(-\cancel{b e^{-\lambda b}}^{\,0} + \left[-\frac{1}{\lambda} \cancel{e^{-\lambda b}}^{\,0} + \frac{1}{\lambda} \right] \right)$$

$$= \frac{1}{\lambda} \quad //$$

Integrals of Unbounded Functions

$\int_{-\infty}^{\infty}$ previously

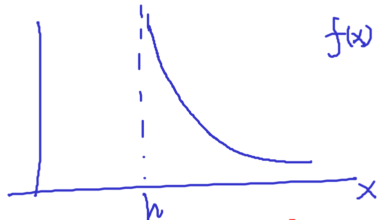
$x \to \infty$

$f(x) \to 0$ as $x \to \infty$

Improper integrals where integrand is not bounded.

$f(x) \to \infty$ as $x \to 0^+$

f(x) is not bounded in y direction



Case 1: f is continuous function in (a, b]. f(x) is not defined at x = a.

e.g. $a=1$, $h = 0.1 \to 0$, 0.01, 0.001, 0.0001 $\downarrow 0^+$

$\underline{a+h}$

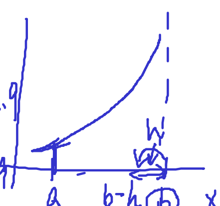
$$\int_{a}^{b} f(x)\,dx = \lim_{h \to 0^+} \int_{\boxed{a+h}}^{b} f(x)\,dx$$

approach a from above. $a+h$

Case 2: f is continuous function in [a, b) but f(x) is not defined at x = b.

e.g. $b=3$, $h = 0.1$, 0.01, 0.001 $\to 0$

$b-h = 2.9$, $= 2.99$, $= 2.999$

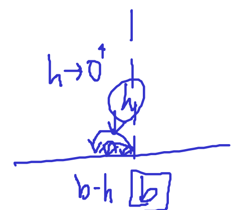


$x=3 \to x = 2.9, 2.99$

$$\int_{a}^{b} f(x)\,dx = \lim_{h \to 0^+} \int_{a}^{b-h} f(x)\,dx$$

0.1, $3-h$

$h \to 0^+$



Case 3: f is continuous in (a, b), but f(x) is not defined at x = a or/and x = b.

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$$

$$= \lim_{h \to 0} \int_{a+h}^{c} f(x)\,dx + \lim_{h \to 0} \int_{c}^{b-h} f(x)\,dx$$

e.g. $For\ f(x) = \frac{1}{\sqrt{x}}\quad x \in (0, 2], solve \int_{0}^{2} f(x)dx.$

$$\lim_{h \to 0^+} \int_{h}^{2} \frac{1}{\sqrt{x}}\,dx = \lim \left(\frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} \Big|_{h}^{2} \right) = \lim\ 2x^{\frac{1}{2}} \Big|_{h}^{2}$$

$\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$, h: $a+h$, $a = 0$

$$= \lim_{h \to 0^+} \left[2\sqrt{2} - 2\sqrt{h} \right]$$

$$= 2\sqrt{2}$$

e.g. *For* $f(x) = \frac{\ln x}{\sqrt{x}}$ *, solve* $\int_0^1 f(x)dx$.

solved today earlier under Int. by sub. indefinite integral

$$\lim_{h\to 0^+} \underbrace{\int_h^1 \frac{\ln x}{\sqrt{x}} dx}_{} = \lim_{h\to 0^+} \left(2\sqrt{x}\ln x - 4\sqrt{x} \Big|_h^1 \right)$$

$u=\sqrt{x}$

$$= \lim_{h\to 0^+} \left((2\sqrt{1}\ln 1 - 4\sqrt{1}) - (2\sqrt{h}\ln h - 4\sqrt{h}) \right)$$

$$= \lim_{h\to 0} \left(-4 - \cancel{2\sqrt{h}\ln h} + \cancel{4\sqrt{h}} \right)$$

$$= -4$$ //

Differentiation with respect to the limits of integration.

upper bound

$$\frac{d}{dt}\int_a^t f(x)dx$$

way 1 solve $\int$ → take $\frac{d}{dt}$

way 2 $f(t)$

$$= \frac{d}{dt}\left[F(t) - F(a) \right]$$

$$= F'(t) = \underline{f(t)}$$

integrand evaluated at $x=t$

lower bound $\frac{d}{dt}\int_t^b f(x)dx \longrightarrow$ way 2 $-f(t)$?

$$= \frac{d}{dt}\left[F(b) - F(t) \right]$$

$$= -f(t)$$

simpler ex. $\frac{d}{dt}\int_2^t \frac{1}{x-1}dx$

In general

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x)\,dx = \frac{d}{dt}\left[F(b(t)) - F(a(t)) \right]$$

$$= F'(b(t))\cdot b'(t) - F'(a(t))\cdot a'(t)$$

$$= \boxed{f(b(t))\cdot b'(t) - f(a(t))\cdot a'(t)}$$

e.g. $\frac{d}{dt}\underbrace{\int_{2t}^{10t} \overset{f(x)}{\frac{1}{x-1}}\, dx}_{I}$

$$I = \ln(x-1)\Big|_{2t}^{10t}$$

$$= \underset{F(b(t))}{\ln(10t-1)} - \underset{F(a(t))}{\ln(2t-1)}$$

$$\frac{d}{dt}\left[\ln(10t-1) - \ln(2t-1)\right]$$

$$= \underbrace{\frac{1}{10t-1}}_{f(10t)} \cdot \underset{\frac{d}{dt}(10t)}{10} - \underbrace{\frac{1}{2t-1}}_{f(2t)} \cdot \underset{\frac{d}{dt}(2t)}{2} = \frac{10}{10t-1} - \frac{2}{2t-1}$$ //

without taking $\int$ & $\frac{d}{dt}$, directly solve it as

$$\left(\frac{1}{10t-1}\right)\cdot 10 - \left(\frac{1}{2t-1}\right)\cdot 2$$

$\frac{1}{10t-1} = f(b(t))$, $10 = b'(t) = \frac{d}{dt}(10t)$, $\frac{1}{2t-1} = f(a(t))$, $2 = a'(t) = \frac{d}{dt}(2t)$ //