ECO240 HW1 Answers

Q1 (6.1) a.

Die outcome	Probability
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

b. Sampling Distribution of the sample means.

Total	sample	$\bar{X}$	Prob.($\bar{X}$)
2	(1,1)	1	1/36
3	(1,2) (2,1)	1.5	2/36
4	(1,3) (3,1) (2,2)	2	3/36
5	(1,4) (4,1) (2,3) (3,2)	2.5	4/36
6	(1,5) (5,1) (2,4) (4,2) (3,3)	3	5/36
7	(1,6) (6,1) (2,5) (5,2) (3,4) (4,3)	3.5	6/36
8	(2,6) (6,2) (3,5) (5,3) (4,4)	4	5/36
9	(3,6) (6,3) (4,5) (5,4)	4.5	4/36
10	(4,6) (6,4) (5,5)	5	3/36
11	(5,6) (6,5)	5.5	2/36
12	(6,6)	6	1/36

Q2 (6.6) $\mu = 100$, $\sigma^2 = 900$, $n = 30$

a. $E(\bar{X}) = \mu = 100$

This is a small sample case (n is 3% of N)

$$\sigma^2_{\bar{x}} = \left(\frac{\sigma}{\sqrt{n}}\right)^2 = \left(\frac{\sqrt{900}}{\sqrt{30}}\right)^2 = \frac{900}{30} = 30$$

$$\sigma_{\bar{x}} = \sqrt{30}$$

b. $P(\bar{X} > 109) = P\left(\frac{\bar{X}-\mu}{\sigma_{\bar{x}}} > \frac{109-100}{\sqrt{30}}\right) = P(Z > 1.64)$

$= 1 - \underbrace{P(Z < 1.64)}_{\text{Use standard normal table}} = 1 - 0.9495 = 0.0505.$

c. $P(96 \le \bar{X} \le 110) = P\left(\frac{96-100}{\sqrt{30}} \le Z \le \frac{110-100}{\sqrt{30}}\right)$

$= P(-0.73 \le Z \le 1.83) = P(Z \le 1.83) - P(Z \le -0.73)$

$= P(Z \le 1.83) - [1 - P(Z \le 0.73)]$

$= 0.9664 - 1 + 0.7673 = 0.7337$

d. $P(\bar{X} \le 107) = P\left(Z \le \frac{107-100}{\sqrt{30}}\right) = P(Z \le 1.28) = 0.8997.$

Q3. (6.8) $\mu = 400, \sigma^2 = 1600, n = 35$

a. $E(\bar{X}) = \mu = 400$

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{40}{\sqrt{35}} = 6.76$

This is "small sample case". n is 3.5% of N.

$\sigma_{\bar{x}}^2 = \frac{1600}{35} = 45.71$

b. $P(\bar{X} > 412) = P\left(Z > \frac{412-400}{6.76}\right) = P(Z > 1.78)$

$= 1 - P(Z < 1.78) = 1 - 0.9625 = 0.0375.$

c. $P(393 < \bar{X} < 407) = P\left(\frac{393-400}{6.76} < Z < \frac{407-400}{6.76}\right)$

$= P(-1.04 < Z < 1.04) = P(Z < 1.04) - P(Z < -1.04)$

$= P(Z < 1.04) - [1 - P(Z < 1.04)]$

$= 2 \times 0.8508 - 1 = 0.7016.$

d. $P(\bar{X} < 389) = P\left(Z < \frac{389-400}{6.76}\right) = P(Z < -1.63)$

$= 1 - P(Z < 1.63) = 1 - 0.9484 = 0.0516.$

HW1 3

Q4 (6.10) $\mu = 1200$, $\sigma = 400$. $n = 9$.

a. $E(\bar{X}) = \mu = 1200$

b. $\sigma_{\bar{X}}^2 = \frac{\sigma^2}{n} = \frac{(400)^2}{9} = 17{,}778$

c. $\sigma_{\bar{X}} = \sqrt{\sigma_{\bar{X}}^2} = 133.33$

d. $P(\bar{X} < 1050) = P(Z < \frac{1050 - 1200}{133.33}) = P(Z < -1.13) =$

$= 1 - P(Z < 1.13) = 1 - 0.8708 = 0.1292$

Q5 (6.14) $\mu = 87$, $\sigma = 22$, $n = 16$

a. $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{22}{4} = 5.5$

b. $P(\bar{X} < 100) = P(Z < \frac{100 - 87}{5.5}) = P(Z < 2.36) = 0.9909$

c. $P(\bar{X} > 80) = P(Z > \frac{80 - 87}{5.5})$ $P(Z > -1.27) = P(Z < 1.27)$

$= 0.898$

d. $1 - P(85 < \bar{X} < 95) = 1 - P(\frac{85 - 87}{5.5} < Z < \frac{95 - 87}{5.5}) = 1 - P(-0.36 < Z < 1.45)$

$= 1 - [P(Z < 1.45) - P(Z < -0.36)] = 1 - P(Z < 1.45) + 1 - P(Z < 0.36)$

$= 2 - 0.9265 - 0.6406 = 0.4329$

Q6 (6.17) $\sigma = 8$, $n = 4$

$\sigma_{\bar{X}} = \frac{8}{\sqrt{4}} = 4$

a. $P(\bar{X} > \mu + 2) = P(Z > \frac{\mu + 2 - \mu}{\sigma_{\bar{X}}}) = P(Z > \frac{2}{4}) = P(Z > 0.5)$

$= 1 - 0.6915 = 0.3085$

b. $P(\bar{X} < \mu - 3) = P(Z < \frac{(\mu - 3) - \mu}{\sigma_{\bar{X}}}) = P(Z < \frac{-3}{4}) = P(Z < -0.75)$

$= 1 - P(Z < 0.75) = 1 - 0.7734 = 0.2266$

c. $1 - P(\mu - 4 < \bar{X} < \mu + 4) = 1 - P(-\frac{4}{4} < Z < \frac{4}{4}) = 1 - P(-1 < Z < 1)$

$= 1 - [P(Z < 1) - P(Z < -1)] = 1 - [P(Z < 1) - [1 - P(Z < 1)]]$

$= 2 - 2 * P(Z < 1) = 2 - 2(0.8413) = 0.3174$

Q7. (6.24) $N = 250$, $n = 50$, $\sigma = 30$

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$$

This is a "large sample" case since n is 20% of N.

$$= \frac{30}{\sqrt{50}}\sqrt{\frac{250-200}{249}} = 3.8023$$

a. $P(\bar{X} > \mu + 2.5) = P\left(z > \frac{2.5}{3.8023}\right) = P(z > 0.657)$

$= 1 - P(z < 0.657) = 1 - 0.7454 = 0.2546$

b. $P(\bar{X} < \mu - 5) = P\left(z < \frac{-5}{3.8023}\right) = P(z < -1.31)$

$= 1 - P(z < 1.31) = 1 - 0.9049 = 0.0951$

c. $1 - P(\mu - 10 < \bar{X} < \mu + 10)$

$= 1 - P\left(\frac{-10}{3.8023} < z < \frac{10}{3.8023}\right) = 1 - P(-2.63 < z < 2.63)$

$= 1 - [P(z < 2.63) - P(z < -2.63)] = 1 - P(z < 2.63) + (1 - P(z < 2.63))$

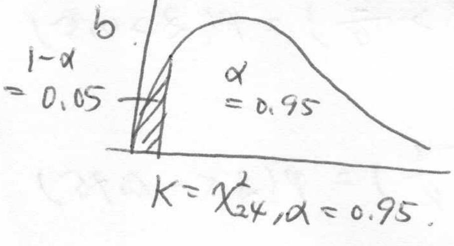
$= 2 - 2 * P(z < 2.63) = 2 - 2 * 0.9957 = 0.0086$

Q8 (6.48) $\mu = 198$, $\sigma^2 = 100$, $n = 25$. $\sigma_{\bar{X}} = \frac{10}{\sqrt{25}} = 2$.

a. $P(\bar{X} > 200) = P\left(z > \frac{200-198}{2}\right) = P(z > 1) = 1 - P(z < 1)$

$= 1 - 0.8413 = 0.1587$

b

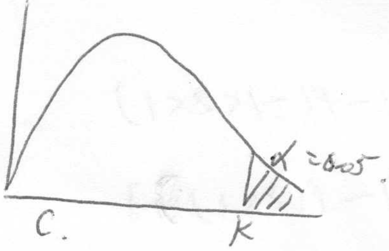


$P(s^2 < A) = P\left(\frac{(n-1)s^2}{\sigma^2} < \frac{(n-1)A}{\sigma^2}\right)$

$= P\left(\chi^2_{24} < \frac{24A}{100}\right) = 0.05$

$\frac{24A}{100} = \chi^2_{24;0.95} = 13.85$

$\boxed{A = \frac{1385}{24} = 57.71}$



c.

$P(s^2 > B) = P\left(\frac{(n-1)s^2}{\sigma^2} > \frac{(n-1)B}{\sigma^2}\right) = P\left(\chi^2_{24} > \frac{24B}{100}\right) = 0.05$

$\frac{24B}{100} = \chi^2_{24,0.05} = 36.42 \Rightarrow$ $\boxed{B = 151.75.}$

Q9. (6.52) $\sigma = 2500$, $n = 16$.

a. $P(S > 3000) = P(S^2 > (3000)^2) = P\left(\frac{(n-1)S^2}{\sigma^2} > \frac{(n-1)(3000)^2}{(2500)^2}\right)$

$= P(\underbrace{\chi^2_{15} > 21.6}) =$ greater than 0.1 (since for $\alpha = 0.1$, the value is 22.31)

└ look for the closest value in the χ^2 table with degrees of freedom $(\nu) = 15$.

b. $P(S < 1500) = P(S^2 < (1500)^2) = P\left(\frac{(n-1)S^2}{\sigma^2} < \frac{(n-1)(1500)^2}{(2500)^2}\right)$

$= P(\chi^2_{(15)} < 5.4)$ ~~between~~

$= 1 - \underbrace{P(\chi^2_{(15)} > 5.4)}_{\text{between } 5.23 \text{ and } 6.26 \rightarrow 0.990 \text{ \& } 0.975} =$ between 0.01 and 0.025.

Q10. (6.54) $\sigma = 100$, $n = 25$

a. $P(S < 75) = P\left(\frac{(n-1)S^2}{\sigma^2} < \frac{(n-1)(75)^2}{\sigma^2}\right)$

$= P\left(\chi^2_{(24)} < \frac{24 \cdot (75)^2}{(100)^2}\right) = P(\chi^2_{(24)} < 13.5)$

$= 1 - \underbrace{P(\chi^2_{(24)} > 13.5)}_{\text{between } 0.975 \text{ \& } 0.95} =$ between 0.025 & 0.05.

b. $P(S > 150) = P\left(\chi^2_{(24)} > \frac{24(150)^2}{(100)^2}\right) = P(\chi^2_{(24)} > 54)$

$=$ less than 0.005.

Q11. (6.55) $\sigma = 4.5$, $n = 30$.

a. $P(S > 3.5) = P\left(\chi^2_{(29)} > \frac{29 * (3.5)^2}{(4.5)^2}\right) = P(\chi^2_{(29)} > 17.54)$

$=$ between 0.975 and 0.95.

→ Yes, more than 0.95.

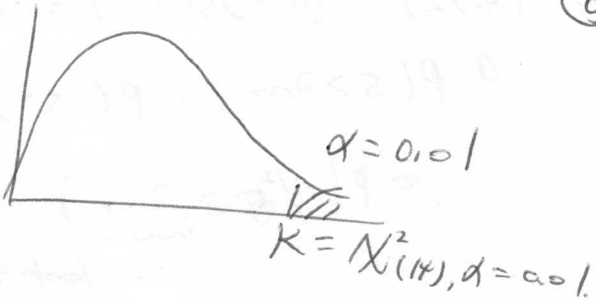
b. $P(S < 6) = P\left(\chi^2_{(29)} < \frac{29 * (6)^2}{(4.5)^2}\right) = P(\chi^2_{(29)} < 51.56)$

$= 1 - \underbrace{P(\chi^2_{(29)} > 51.56)}_{\text{between } 0.01 \text{ and } 0.005}$

$=$ between 0.99 and 0.995 → Yes, more than 0.95.

Q12. (6.59) $n = 15, \ \sigma = 1.8$

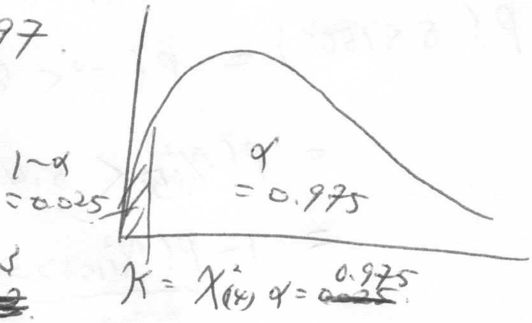
a. $P\left(\chi^2_{(14)} > \dfrac{14(A)^2}{(1.8)^2}\right) = 0.01$



$$\frac{14A}{(1.8)^2} = \chi^2_{(14), 0.01} = 29.14$$

$$(A)^2 = 6.74 \qquad A = 2.597$$

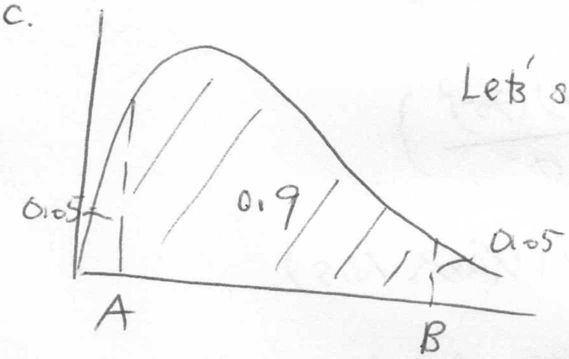
b. $P\left(\chi^2_{(14)} < \dfrac{14(B)^2}{(1.8)^2}\right) = 0.025$



$$\frac{14(B)^2}{(1.8)^2} = \chi^2_{(14)\,0.975} = 5.63$$

$$B^2 = 1.363 \qquad B = 1.141$$

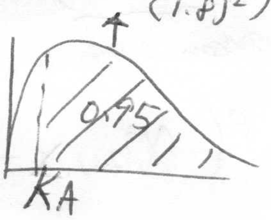
c.

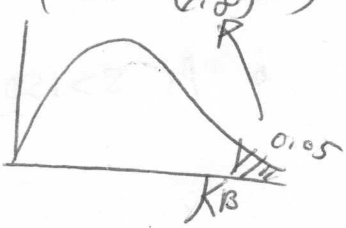


Let's suppose $P(S < A) = 0.05$ and $P(S > B) = 0.05$

$$P(A < S < B) = P\left(\frac{14A^2}{(1.8)^2} < \chi^2_{(14)} < \frac{14B^2}{(1.8)^2}\right) = 0.9$$

$$= P\left(\chi^2_{(14)} > \frac{14A^2}{(1.8)^2}\right) - P\left(\chi^2_{(14)} > \frac{14B^2}{(1.8)^2}\right)$$





$\Rightarrow \dfrac{14A^2}{(1.8)^2} = \chi^2_{(14), \alpha = 0.95} = 6.57$

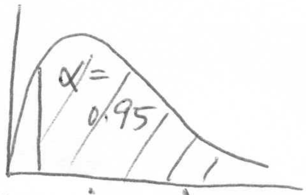
$A^2 = 1.52$, $\boxed{A = 1.23}$

$\Rightarrow \dfrac{14B^2}{(1.8)^2} = \chi^2_{(14)\, \alpha = 0.05} = 23.68$

$B^2 = 5.48 \rightarrow \boxed{B = 2.34}$

Q13. (6.60) $n=12$

a. $P(S^2 > \beta\sigma^2) = 0.95$



$P\left(\chi^2_{(11)} > \frac{(n-1)\beta\sigma^2}{\sigma^2}\right) = P\left(\chi^2_{(11)} > 11\beta\right) = 0.95.$

$11\beta = \chi^2_{(11)\alpha=0.95} = 4.57.$

$\beta = 0.415.$ $\quad$ 41.5%

b. $P(S^2 > \beta\sigma^2) = 0.90$

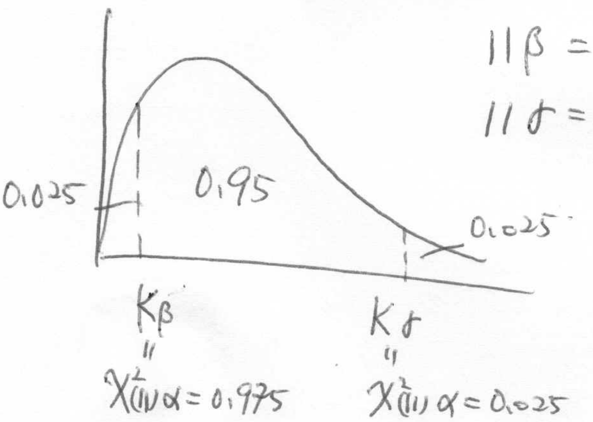
$P\left(\chi^2_{(11)} > \frac{(n-1)\beta\sigma^2}{\sigma^2}\right) = 0.90$

$P\left(\chi^2_{(11)} > 11\beta\right) = 0.90$

$11\beta = \chi^2_{(11)\alpha=0.9} = 5.58$, $\beta = 0.5073$ $\quad$ 50.73%

c. $P(\beta\sigma^2 < S^2 < \gamma\sigma^2) = 0.95.$

$P\left(11\beta < \chi^2_{(11)} < 11\gamma\right) = P\left(\chi^2_{(11)} > 11\beta\right) - P\left(\chi^2_{(11)} > 11\gamma\right)$



$11\beta = \chi^2_{11,0.975} = 3.82 \rightarrow \beta = 0.347$

$11\gamma = \chi^2_{11,0.025} = 21.92. \rightarrow \gamma = 1.993.$

" The probability is 0.95 that the sample variance is between 34.7% and 199.3% of the population variance ".

Q14. $E[(\bar{X}-\mu)^2] = \frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right)$

$\rightarrow E\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right] = n\sigma^2 - \frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right)\cdot n$

$= \frac{n\sigma^2(N-1) - \sigma^2(N-n)}{N-1}$

$= \frac{n\sigma^2 N - n\sigma^2 - N\sigma^2 + n\sigma^2}{N-1}$

$E(S^2) = \frac{1}{n-1}\,\frac{N\sigma^2(n-1)}{N-1} = \frac{N\sigma^2}{N-1}.$ //